Finitely generated, non-artinian monolithic modules.

Ian M. Musson

Department of Mathematical Sciences
University of Wisconsin-Milwaukee
email: `musson@uwm.edu`

January 19, 2013

Abstract

We survey noetherian rings A over which the injective hull of every simple module is locally artinian. Then we give a general construction for algebras A that do not have this property. In characteristic 0, we also complete the classification of down-up algebras with this property which was begun in [CLPY10] and [CM].

1 Introduction

A module M is *monolithic* if the intersection of all nonzero submodules of M is nonzero. The intersection of all nonzero submodules of a monolithic module M is a simple submodule known as the *lith* of M. Thus monolithic modules have a unique lith! This terminology is due to Roseblade [Ros73], [Ros76]. It was pointed out to me by Ken Goodearl that monolithic modules are also known as subdirectly irreducible modules. We consider the following property of a noetherian ring A.

    ($\diamond$)   Every finitely generated monolithic A-module is artinian.

Equivalently, the injective hull of every simple A-module is locally artinian. Some history concerning property ($\diamond$) is given in the introduction to [CM]. The property is not well understood, as is shown by the following quite baffling lists of examples.

The following rings A have property ($\diamond$).

(A.0) Commutative noetherian rings, and more generally PI and FBN rings [Jat74b].

The next two examples are in fact PI rings.

(A.1) The coordinate ring of the quantum plane, that is the algebra generated by elements a, b subject to the relation $ab = qba$ when $q \in K$ is a root of unity.

(A.2) The quantized Weyl algebra, that is the algebra generated by elements a, b subject to the relation $ab - qba = 1$ when $q \in K$ is a root of unity.

(A.3) The enveloping algebra $U(\mathfrak{sl}(2, K))$ where K a field of characteristic 0, [Dah84].

(A.4) The group rings $\mathbb{Z}G$ and KG where K is a field which is algebraic over a finite field and G is polycyclic-by-finite, [Jat74], [Ros76].

(A.5) Prime noetherian rings of Krull dimension 1, [CLPY10], [Mus80].

(A.6) There are simple noetherian, non-artinian rings for which any simple module is injective, and obviously these rings have property ($\diamond$) [Coz70].

The following rings A do not have property ($\diamond$).

(B.1) The coordinate ring of the quantum plane when $q \in K\backslash\{0\}$ is not a root of unity, [CM].

(B.2) The quantized Weyl algebra, when $q \in K\backslash\{0\}$ is not a root of unity, [CM].

(B.3) The enveloping algebra $U(\mathfrak{b})$ over an algebraically closed field of characteristic 0, when $\mathfrak{b}$ is finite dimensional, solvable and non-nilpotent, [CH80], [Mus82].

(B.4) The group algebra KG where K is a field which is not algebraic over a finite field and G is polycyclic-by-finite which is not nilpotent-by-finite, [Mus80].

(B.5) The Goodearl-Schofield example: a certain non-prime noetherian ring of Krull dimension 1, [GS86].

What has been lacking up to now is a general construction for finitely generated, non-artinian, monolithic modules. In the next section we give such a construction under fairly mild conditions on A. We show that examples (B.1)-(B.3) satisfy these conditions. We also apply our construction to down-up algebras in characteristic 0. Some open problems are given in the last section.

2   The construction.

Let K be a field. We make the following assumptions.

(1) A is a noetherian K-algebra without zero divisors.

(2) w is a normal element of A.

(3) J is a maximal left ideal such that $w - \mu \in J$ for some non-zero $\mu \in K$.

From (1) and (2) it follows that there is an automorphism σ of A such that for any $x \in A$ we have

$$wx = \sigma(x)w. \tag{2.1}$$

Suppose that x is an element of A that is not a unit and set $I = Jx$. Then we have a short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

where $L = Ax/I, M = A/I$ and $N = A/Ax$.

Lemma 2.1. $L \cong A/J$ *is a simple A-module.*

Proof. The map f from A to L sending a to $ax + Jx$ is clearly surjective with kernel containing J. If $a \in \operatorname{Ker} f$, then $(a - j)x = 0$ for some $j \in J$, whence $a \in J$. $\square$

An interesting feature of our construction is that remaining assumptions involve only L and N. There is a single additional assumption on L.

(4) For all $m \geq 0$ the equation

$$\sigma^m(x)a - 1 \in J \tag{2.2}$$

has no solution for $a \in A$. For $z \in A$, denote the image of z in $M = A/I$ by $\overline{z}$. Then equation (2.2) is equivalent to

$$\sigma^m(x)a\overline{x} = \overline{x} \tag{2.3}$$

and equation (2.3) always has a solution if L is divisible. Since L obviously cannot be injective, some condition similar to (4) must be necessary if our construction is to go through.

Finally we make the following assumptions on N.

(5) N has a strictly descending chain of submodules

$$N \supset wN \supset \ldots \supset w^m N \supset \ldots \tag{2.4}$$

(6) Every nonzero submodule of N contains $w^m N$ for some m.

Theorem 2.2. *Under assumptions (1)-(6), M is an essential extension of L.*

Proof. Note that the assumptions are unchanged if we replace w by $\mu^{-1}w$. Thus we can assume that $\mu = 1$. Suppose U is a left ideal of A strictly containing I. We need to show that U contains Ax. It follows easily from (6) that U contains an element of the from $w^m - ax$ for some $a \in A$. Set $y = \sigma^m(x)$. Then from (2.1) and (3) we have

$$\begin{aligned} y(w^m - ax) &= (w^m - 1)x + (1 - ya)x \\ &\equiv (1 - ya)x \mod Jx. \end{aligned} \tag{2.5}$$

For $z \in A$, denote the image of z in $M = A/I$ by $\overline{z}$. From (2.5) and assumption (4) we have $0 \neq (1 - ya)\overline{x} \in \overline{U} \cap A\overline{x}$, so as $L = A\overline{x}$ is simple it follows that $A\overline{x} \subseteq \overline{U}$. The result follows easily. $\square$

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3 Examples (B.1)-(B.3).

To check assumption (4) we use the following easy result.

Lemma 3.1. *If for all $m \geq 0$, there is a subring B of A such that $A = B \oplus J$, and $\sigma^m(x) \in B$, then assumption (4) holds.*

Proof. If $\sigma^m(x)a - 1 \in J$, write $a = b + j$ with $b \in B$ and $j \in J$. Then $\sigma^m(x)b - 1 \in J \cap B = 0$, whence $\sigma^m(x)$ is a unit in A a contradiction, since x is assumed to be a non-unit. $\qquad\square$

It is not always possible to choose B to be σ-invariant in Lemma 3.1. From Theorem 2.2 and the next two results, we obtain the non-artinian, monolithic modules in [CM] Theorems 3.1 and 4.2.

Let $A = K[a, b]$ be the coordinate ring of the quantum plane, as in (B.1) where $ab = qba$ and $q \in K \backslash \{0\}$ is a not root of unity. Let $w = ab$ and $J = A(ab - 1)$, $B = K[a]$ and $x = a - 1 \in B$. Then w is a normal element and the automorphism σ determined by equation (2.1) satisfies $\sigma(a) = q^{-1}a$ and $\sigma(b) = qb$.

Proposition 3.2.

(a) J is a maximal left ideal of A and assumption (4) holds.

(b) If $N = A/Ax$, then N is non-artinian, and a complete list of non-zero submodules of N is given by equation (2.4).

Proof. Since $A = B \oplus J$ and σ preserves B, the result follows from Steps 1 and 2 in the proof of [CM] Theorem 3.1. $\qquad\square$

Let $A = K[a, b]$ be the quantized Weyl algebra, as in (B.2) where $ab - qba = 1$ and $q \in K \backslash \{0\}$ is a not root of unity. If $w = ab - ba$, then w is a normal element of A and $w - 1 = (q - 1)ba \in J = Aa$. The automorphism σ determined by equation (2.1) satisfies $\sigma(a) = q^{-1}a$ and $\sigma(b) = qb$. We have $A = B \oplus J$ with $B = K[b]$, and $\sigma(B) = B$. Let $x = (1 - q)b - 1 \in B$.

Proposition 3.3.

(a) J is a maximal left ideal of A and assumption (4) holds.

(b) If $N = A/Ax$, then N is non-artinian, and a complete list of non-zero submodules of N is given by equation (2.4).

Proof. By [CM] Lemma 4.1, J is a maximal left ideal of A, and (4) follows as before. Note that $N \cong K[a]$ as a $K[a]$-module. Let $u_0 = 1 + Ax$, and define inductively $u_{n+1} = (q^{-n}a - 1)u_n$. Then

$$au_n = q^n(u_n + u_{n+1}) \quad \text{and} \quad bu_n = \frac{q^{-n}}{1 - q}u_n.$$

Thus (b) follows as in the proof of [CM] Theorem 4.2 (b). $\qquad\square$

Next we show that certain Ore extensions with Gelfand-Kirillov dimension 2 do not have property ($\diamond$). Assume that K has characteristic zero, and let d be the derivation of the polynomial algebra $K[a]$ determined by $d(a) = a^r$ where $r \geq 1$. Let $A = K[a, b]$ be the resulting Ore extension, where for $p \in K[a]$,

$$pb = bp + d(p). \tag{3.1}$$

In particular

$$ab = ba + a^r.$$

Thus if $w = a$, then w is a normal element and the automorphism σ determined by equation (2.1) satisfies $\sigma(a) = a$ and $\sigma(b) = b + a^{r-1}$. We show below that A does not have property ($\diamond$). When $r = 1$, A is isomorphic to he enveloping algebra $U(\mathfrak{b})$ where, $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{sl}(2, K)$. Now by [BGR73] Lemma 6.12, if K is algebraically closed, then any finite dimensional solvable Lie algebra which is non-nilpotent has $\mathfrak{b}$ as an image, and thus we recover the result in (B.3).

Lemma 3.4. *Any ideal invariant under d is generated by a power of a.*

Proof. This follows from the well known fact that if an ideal Q is invariant under a derivation, then so too are all the prime ideals that are minimal over Q, see for example [BGR73] Lemma 4.1. $\square$

Let $J = A(a - 1)$ and $x = b - 1$.

Proposition 3.5.

(a) J is a maximal left ideal of A and assumption (4) holds.

(b) If $N = A/Ax$, then N is non-artinian, and a complete list of non-zero submodules of N is given by equation (2.4).

(c) The submodules of N are pairwise non-isomorphic.

Proof. (a) Set $v_n = b^n + J$. The elements $\{v_n\}_{n \geq 0}$ form a basis for A/J, and $av_0 = v_0$. Assume by induction that

$$(a - 1)^n v_n = n! v_0. \tag{3.2}$$

Then by equation (3.1), we have

$$
\begin{aligned}
(a - 1)^{n+1} v_{n+1} &= (a - 1)^{n+1} b v_n \\
&= b[(a - 1)^{n+1} + (n + 1)a^r (a - 1)^n] v_n \\
&= (n + 1)! v_0
\end{aligned}
$$

It follows easily from equation (3.2) that A/J is simple. Since $\sigma^m(x) = b - 1 + ma^{r-1}$ we have $A = B \oplus J$ where $B = K[\sigma^m(x)]$, thus (4) holds.

(b) Since $A = K[a] \oplus Ax$, we can identify N with $K[a]$ as a $K[a]$-module. Suppose N' is a submodule of N, and $N' = pK[a]$ for some $p \in K[a]$. Then

$$
\begin{aligned}
bp &= pb - d(p) \\
&\equiv p - d(p) \mod Ax,
\end{aligned}
$$

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and hence $d(p) \in pK[a]$. Thus (b) follows from Lemma 3.4.

(c) As above we identify N with $K[a]$. If $\phi : a^m N \longrightarrow a^{m_1} N$ is an isomorphism, then $\phi(a^m) = a^{m_1} q(a)$ for some polynomial q with $q(0) \neq 0$. Thus

$$
\begin{aligned}
\phi(ba^m) &= \phi(a^m - ma^{m+r-1}) \\
&= (1 - ma^{r-1})a^{m_1} q(a).
\end{aligned}
$$

and

$$
\begin{aligned}
b\phi(a^m) &= b(a^{m_1} q(a)) \\
&= a^{m_1} q(a) - a^r (a^{m_1} q(a))'.
\end{aligned}
$$

This easily gives

$$
(m_1 - m)a^{m_1+r-1} q(a) = a^{m_1+r} q'(a).
$$

Now we must have $m = m_1$ since otherwise the left side has 0 as a root of multiplicity at most $m - r + 1$, whereas the right side has 0 as a root of multiplicity at least $m - r$. $\qquad\blacksquare$

4   Down-up Algebras.

Given a field K and α, β, γ elements of K, the associative algebra $A = A(\alpha, \beta, \gamma)$ over K with generators d, u and defining relations

$$
(R1) \qquad d^2 u = \alpha dud + \beta u d^2 + \gamma d
$$

$$
(R2) \qquad du^2 = \alpha udu + \beta u^2 d + \gamma u
$$

is called a down-up algebra. Down-up algebras were introduced by G. Benkart and T. Roby [BR98], [BR99]. In [KMP99] it is shown that $A = A(\alpha, \beta, \gamma)$ is noetherian if and only if $\beta \neq 0$, and that these conditions are equivalent to A being a domain. The main result of this section is as follows.

Theorem 4.1. *If $A(\alpha, \beta, \gamma)$ is a noetherian down-up algebra over a field K of characteristic zero, then any finitely generated monolithic $A(\alpha, \beta, \gamma)$-module is artinian if and only if the roots of $X^2 - \alpha X - \beta$ are roots of unity.*

From now on we assume that $X^2 - \alpha X - \beta = (X - 1)(X - \eta)$ where $\eta = -\beta$ is not a root of 1, and that $\beta \neq 0$. Thus $A(\alpha, \beta, \gamma)$ is a Noetherian domain by the above remarks, and $\alpha + \beta = 1$. In addition we assume that $\gamma \neq 0$. Hence $A(\alpha, \beta, \gamma)$ is isomorphic to a down-up algebra

$$
A_\eta = A(1 + \eta, -\eta, 1).
$$

To prove Theorem 4.1 it is enough to prove the result below, as noted in [CM].

Theorem 4.2. *If η is not a root of unity, then A_η does not have property $(\diamond)$.*

For the remainder of this section we assume that $A = A_\eta$ as in Theorem 4.2. We begin with some consequences of (R1) and (R2). Since $\eta \neq 1$, we have $\alpha \neq 2$. Set $\epsilon = (\alpha - 2)^{-1}$, and $\phi = 1 - \alpha\epsilon = -2(\alpha - 2)^{-1}$. As noted in [CM00] Section 1.4 Case 2, the element $w = -ud + du + \epsilon$ satisfies

$$dw = \eta wd, \quad uw = \eta^{-1} wu,$$

and hence $wA = Aw$. We remark that A/Aw is isomorphic to the first Weyl algebra (this fact is not used below).

Lemma 4.3. *For $n \geq 1$, we have*

$$du^{2n} = u^{2n}d + n\phi u^{2n-1} + \alpha \sum_{i=0}^{n-1} \eta^{-2i-1} wu^{2n-1} \tag{4.1}$$

and for $n \geq 0$,

$$du^{2n+1} = u^{2n+1}d + u^{2n}w + (n\phi - \epsilon)u^{2n} + \alpha \sum_{i=0}^{n-1} \eta^{-1-2i} wu^{2n}. \tag{4.2}$$

Proof. We have
$$du = w + ud - \epsilon. \tag{4.3}$$

Using (R2), then (4.3) and the fact that $\alpha + \beta = 1$, we see that for $j \geq 2$,

$$
\begin{aligned}
du^j &= [\alpha u du + \beta u^2 d + u]u^{j-2} \\
&= [\alpha u(w + ud - \epsilon) + \beta u^2 d + u]u^{j-2} \\
&= [(\alpha + \beta)u^2 d + \alpha uw + (1 - \alpha\epsilon)u]u^{j-2} \\
&= u^2 du^{j-2} + \alpha uwu^{j-2} + \phi u^{j-1}.
\end{aligned}
$$

The result follows easily by induction. $\qquad\square$

Consider the module $N = A/A(d-1)$, and if $a \in A$, denote the image of a in N by $\bar{a}$. Then N has a basis $w^i \bar{u}^j$ with $i, j \geq 0$. Thus if $B = K[u, w]$, then $N \cong B$ as a left B-module. Since $dw^m = \eta^m w^m d$, N has a strictly descending chain of submodules as in Assumption (5). Next we define a filtration on N by setting

$$N_n = \sum_{i=0}^{n} u^i K[\bar{w}] = \sum_{i=0}^{n} K[w]\bar{u}^i.$$

It follows from (4.1) and (4.2) that $dN_n \subseteq N_n$. Also for $f \in K[w]$, we have

$$df(w)\bar{u}^n \equiv f(\eta w)\bar{u}^n \mod N_{n-1}. \tag{4.4}$$

Lemma 4.4. *If U is a non-zero submodule of N, then U contains $\bar{w}^m$ for some m.*

Proof. Suppose that n is minimal such that $U \cap N_n \neq 0$. We claim that $n = 0$. If this is not the case then $U + N_{n-1}$ contains an element of the form $x = f(w)\overline{u}^n$ for some non-zero polynomial f. Write $f(w) = \sum_{i=r}^{s} a_i w^i$, where $a_r \neq 0 \neq a_s$. If $r < s$, then $U + N_{n-1}$ contains an element of the form $y = w^r \overline{u}^n$, because $\prod_{i=r+1}^{s}(d - \eta^i)x \in U + N_{n-1}$. Thus if $n = 2m$ is even, we can assume that

$$y = w^r \overline{u}^{2m} + \sum_{i=0}^{2m-1} g_i(w)\overline{u}^i \in U.$$

Then

$$(d-\eta^r)y \equiv [\eta^r w^r(m\phi + \alpha \sum_{i=0}^{n-1} \eta^{-2i-1}w) + g_{2m-1}(\eta w) - \eta^r g_{2m-1}(w)]\overline{u}^{2m-1} \quad \text{mod } N_{n-2}.$$

By the choice of $n, (d - \eta^r)y$ must be zero mod N_{n-2}. Note that the coefficient of w^r in $g_{2m-1}(\eta w) - \eta^r g_{2m-1}(w)$ is zero. Thus looking at the coefficient of $w^r \overline{u}^{2m-1}$ on the right side above yields $m\phi = 0$, which is a contradiction. Thus $n = 2m + 1$ is odd, and we can assume that

$$y = w^r \overline{u}^n + \sum_{i=0}^{2m} f_i(w)\overline{u}^i \in U + N_{n-2}.$$

Then mod N_{n-2},

$$
\begin{aligned}
(d - \eta^r)y &\equiv \eta^r w^r[u^{2m}\overline{w} + (m\phi - \epsilon)\overline{u}^{2m} + \alpha \sum_{i=0}^{m-1} \eta^{-1-2i}w\overline{u}^{2m}] \\
&+ [f_{2m}(\eta w) - \eta^r f_{2m}(w)]\overline{u}^{2m}.
\end{aligned}
$$

By the choice of $n, (d-\eta^r)y$ must be zero mod N_{n-2}. Then looking at the coefficient of $w^r \overline{u}^{2m}$ we obtain $m\phi = \epsilon$ which leads to the contradiction $2m + 1 = 0$. Thus U contains an element of the form $f(\overline{w})$ with $f \neq 0$, and the result follows easily. $\square$

We have verified assumptions (5) and (6) for the module N, and we now turn our attention to the simple module L.

Following [BR98] Proposition 2.2, we define the Verma module $V(\lambda)$ with highest weight $\lambda \in K$. Let $\lambda_{-1} = 0, \lambda_0 = \lambda$ and for each $n > 0$ set,

$$\lambda_n = \alpha\lambda_{n-1} + \beta\lambda_{n-2} + 1. \tag{4.5}$$

The Verma module $V(\lambda)$ has basis $\{v_n | n \in \mathbb{N}\}$. The action of A is defined by

$$d.v_0 = 0, \text{ and } d.v_n = \lambda_{n-1}v_{n-1}, \text{for all } n \geq 1$$

$$u.v_n = v_{n+1}.$$

In [BR98] Proposition 2.4 it is shown that $V(\lambda)$ is simple if and only if $\lambda_n \neq 0$ for all $n \geq 0$. Furthermore, by [CM00] Lemma 2.5, $\lambda_{n-1} = 0$ if and only if

$$\lambda(\eta - 1) = -(1 - n(\sum_{i=0}^{n} \eta^i)^{-1}). \tag{4.6}$$

Lemma 4.5. *The algebra A has infinitely many pairwise non-isomorphic simple Verma modules.*

Proof. The result is evident if K is uncountable, because then we simply require that the highest weight λ does not satisfy the condition in (4.6) for any n. In general we argue as follows. By [CM00] Proposition 5.5, any Verma module has length at most 3, so by [BR98] Proposition 2.23, any Verma module has a simple Verma submodule. Also if $V(\lambda)$ is not simple this submodule is generated by v_n where n is the largest integer such that $\lambda_{n-1} = 0$. This submodule is isomorphic to $V(\lambda_n)$. Note that the case covered by [BR99] does not arise here. Now if $\mu = \lambda_n$ and $V(\mu)$ is simple, we can solve the recurrence (4.5) in reverse to find all Verma modules $V(\lambda)$ containing as a $V(\mu)$ simple submodule. Since there can be at most 3 such λ and K is infinite, the result follows. $\qquad\square$

Unfortunately it does not seem possible to verify assumption (4) for a simple Verma module. Instead we consider the universal lowest weight modules $W(\kappa)$ defined in [BR98] Proposition 2.30 (a). For $\kappa \in K$, set $\kappa_{-1} = 0, \kappa_0 = \kappa$ and define for each $n > 0$,

$$\kappa_n = \eta^{-1}(\alpha\kappa_{n-1} - \kappa_{n-2} + 1). \qquad (4.7)$$

Then $W(\kappa)$ has basis $\{a_n | n \in \mathbb{N}\}$. The action of A is defined as follows,

$$u.a_0 = 0, \text{ and } u.a_n = \kappa_{n-1}a_{n-1}, \text{for all } n \geq 1$$

$$d.a_n = a_{n+1}.$$

Corollary 4.6. *The algebra A has infinitely many pairwise non-isomorphic simple lowest weight modules $W(\kappa)$.*

Proof. By [CM00] Lemma 4.1, there is an isomorphism from A onto $A' = A_{\eta^{-1}}$ which interchanges the generators u and d. Under this isomorphism, any Verma module for A' becomes a module of the form $W(\kappa)$ for A, so the result follows. $\qquad\square$

Proof of Theorem 4.2. Let $L = W(\kappa)$ be a simple lowest weight module, and let J be the annihilator of the lowest weight vector a_0 in A. Then $J = Au + A(ud - \kappa)$. The normal element $w = -ud + du + \epsilon$ satisfies $w - \mu \in J$ where $\mu = -\kappa + \epsilon$. By Corollary 4.6 we can arrange that μ is non-zero. Set $x = d - 1$. It only remains to check assumption (4). This holds because $A = B \oplus J$ with $x \in B = K[d]$, and B is σ-invariant. $\qquad\square$

5 Remarks and Problems.

(a) We call a finitely generated module E over a left noetherian ring *uniserial* if the submodules of E are totally ordered by inclusion. For E uniserial define a descending chain of submodules $\{E_\alpha\}$ as follows. For any ordinal α, if $E_\alpha \neq 0$ let $E_{\alpha+1}$ be the unique maximal submodule of E_α. For a limit ordinal β such that $E_\alpha \neq 0$ for $\alpha < \beta$, set $E_\beta = \bigcap_{\alpha < \beta} E_\alpha$. There is a smallest ordinal τ such that $E_\tau = 0$, and we call τ the *depth* of E. As noted in the introduction to

[Jat69], it follows from [Jat69] Theorem 4.6, that for any ordinal τ there is a left noetherian ring A such that the left regular module is uniserial with depth τ. The modules M constructed using Theorem 2.2 with the aid of the results in Section 3 are all uniserial with depth $\omega + 1$ where ω is the first infinite ordinal. What other module depths are possible for uniserial modules over (two-sided) noetherian rings?

(b) If N is as in Propositions 3.2 and 3.3 (resp. 3.5), then N is incompressible and critical by [CM] Theorems 3.1 and 4.2, (resp. Proposition 3.5 (c)). The first example of an a incompressible and critical module was found by Ken Goodearl, see [Goo80], to which we refer for the definitions. In general is there a connection between rings that do not have A property $(\diamond)$, and incompressible critical modules?

(c) Suppose that A is a Noetherian ring, and P an ideal of A such that A/P is simple artinian with simple module S. Is the injective hull of a S as an A-module locally artinian?

(d) Define a noetherian ring A to be $(\diamond)$ *extremal* if it does not have property $(\diamond)$, but every proper homomorphic image has property $(\diamond)$. What can be said about $(\diamond)$ extremal rings? If A is an algebra over a field having finite Gelfand-Kirillov dimension and A is $(\diamond)$ extremal, must A be prime? The Goodearl-Schofield example shows that this is not true without the GK dimension hypothesis. It seems likely that the algebra A_η in Theorem 4.2 is $(\diamond)$ extremal. We note the following result.

Proposition 5.1. *Suppose that A is a K-algebra such that the endomorphism ring of every simple A-module is algebraic over K. If A is $(\diamond)$ extremal the center Z of A is algebraic over K.*

Proof. If Z is not algebraic over K then, for every simple module L, the natural map $Z \longrightarrow \operatorname{End}_A L$ has non-zero kernel $\mathbf{m}$. Then if the injective hull of L as an $A/\mathbf{m}A$ is locally artinian, then so too is its injective hull over A, see [CLPY10] Proposition 1.6. Thus A cannot be $(\diamond)$ extremal. $\square$

The hypothesis that the endomorphism ring of every simple A-module is algebraic over K is known to hold for many algebras, for example it holds for almost commutative algebras (Quillen's Lemma) and for an algebra of countable dimension over an uncountable field.

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